Balances of m-bonacci words

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Abstract

The m-bonacci word is a generalization of the Fibonacci word to the m-letter alphabet $\mathcal{A} = \{0, \ldots, m-1\}$. It is the unique fixed point of the Pisot–type substitution $\varphi_m : 0 \to 01,\ 1 \to 02,\ \ldots,\ (m-2) \to 0(m-1)$, and $(m-1) \to 0$. A result of Adamczewski implies the existence of constants $c^{(m)}$ such that the m-bonacci word is $c^{(m)}$-balanced, i.e., numbers of letter a occurring in two factors of the same length differ at most by $c^{(m)}$ for any letter $a \in \mathcal{A}$. The constants $c^{(m)}$ have been already determined for $m = 2$ and $m = 3$. In this paper we study the bounds $c^{(m)}$ for a general $m \geq 2$. We show that the m-bonacci word is $(\lfloor \kappa m \rfloor + 12)$-balanced, where $\kappa \approx 0.58$. For $m \leq 12$, we improve the constant $c^{(m)}$ by a computer numerical calculation to the value $\lceil \frac{m+1}{2} \rceil$.

1 Introduction

The m-bonacci word is a generalization of the Fibonacci word to the m-letter alphabet $\mathcal{A} = \{0, \ldots, m-1\}$. It is the unique fixed point of the substitution $\varphi = \varphi_m$ given by the prescription

$$0 \to 01,\ 1 \to 02,\ \ldots,\ (m-2) \to 0(m-1),\ \text{and}\ (m-1) \to 0. \tag{1}$$

In particular, for $m = 3$, we obtain the substitution $0 \to 01,\ 1 \to 02,\ 2 \to 0$ with the fixed point

$$0102010010201010201001020102010010201010201001020100102010102010010201020100\cdots,$$

usually called the Tribonacci word.

The aim of this article is to study a certain combinatorial property of the m-bonacci word for a general m. Namely, we examine the balance property, which describes a certain uniformity of occurrences of letters in an infinite word. In order to give its rigorous definition, let us precise the notation we will use in the sequel. A factor of an infinite word $\mathbf{u} = \mathbf{u}_0\mathbf{u}_1\mathbf{u}_2 \cdots \in \mathcal{A}^{\mathbb{N}}$ is any finite string in the form $w = \mathbf{u}_i\mathbf{u}_{i+1} \cdots \mathbf{u}_{i+n-1}$ for certain $i \in \mathbb{N}_0$, $n \in \mathbb{N}$, where $|w| = n$ is the length of the factor w. The language of an infinite word $\mathbf{u}$, denoted by $\mathcal{L}(\mathbf{u})$, is the set of all its factors. The number of occurencies of a given letter $a \in \mathcal{A}$ in a factor w is denoted by $|w|_a$. Clearly, $\sum_{a \in \mathcal{A}} |w|_a = |w|$. The balance property is related to the variability of $|w|_a$ within the meaning of the following definition.

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Definition 1. *Let c be a positive integer. An infinite word $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ is said to be c-balanced if*

$$|w|_a - |v|_a \leq c$$

for all factors $w, v \in \mathcal{L}(\mathbf{u})$ of the same length and for each letter $a \in \mathcal{A}$.

The notion of a 1-balanced word (originally referred to as "balanced word") has been used by Morse and Hedlund already in 1940 [8] for a characterization of Sturmian sequences. Since the Fibonacci word (in our notation 2-bonacci word) is Sturmian, it is 1-balanced.

It was expected and announced in several papers since 2000 that the Tribonacci word is 2-balanced [5, 4, 13]. This statement has been proved in 2009 (in two different ways) by Richomme, Saari and Zamoboni [11]. As for a general $m \geq 2$, in 2009 Glen and Justin [7] mentioned "the k-bonacci word is $(k-1)$-balanced", but to the best of our knowledge, no proof of this proposition has ever been published.

The m-bonacci words belong to a broad class called Arnoux–Rauzy words. In the last ten years, balance properties of Arnoux–Rauzy words have been intensively studied. For the most recent results and a nice overview see [3].

The works of Adamczewski on discrepancy and balance properties of fixed points of primitive substitutions [1, 2] imply the existence of finite constants $c^{(m)}$ such that the m-bonacci word is $c^{(m)}$-balanced. Namely, Adamczewski proved that if all eigenvalues of the matrix of substitution except the dominant one are of modulus less than 1, then the fixed point of the primitive substitution is c-balanced for some c. It is well known (and explicitly shown in our text as well) that the substitution defined by (1) satisfies the Adamczewski condition.

In the present article, we approach the problem of determining $c^{(m)}$ by refining the matrix method used by Adamczewski in [1, 2] (and also by Richomme, Saari, Zamboni in [11] in their Proof 2). Small values of m can be treated numerically. We show that

- the 4-bonacci word and the 5-bonacci word are 3-balanced but not 2-balanced;
- for $m = 6, 7, \ldots, 12$ the m-bonacci word is $\lceil \frac{m+1}{2} \rceil$-balanced, Theorem 3.1.

The approach works for a general m as well. We prove the following theorem.

Theorem. (Theorem 6.1.) *The m-bonacci word is $c^{(m)}$-balanced with*

$$c^{(m)} = \lfloor \kappa m \rfloor + 12,$$

where $\kappa = \frac{2}{\pi} \int_0^{2\pi} \frac{1-\cos x}{(5-4\cos x)\ln(5-4\cos x)} \mathrm{d}x \approx 0.58.$

Our results confirm the bound $c = m - 1$ proposed by Glen and Justin for all $m \leq 12$ and $m \geq 29$. Moreover, it turns out that the formerly proposed bound $c = m - 1$ is far from being optimal except for a few small values of m.

Our article is organized as follows: Section 2 explains relationship between balance and discrepancy and gives a formula estimating the balance constant using spectrum of the matrix M of substitution (1). In Section 3, we present results obtained by computer evaluation of this formula. In Section 4, we show that for estimating the balance constant c we can concentrate on the letter 0 only. Sections 5 and 6 are devoted to the proof of the main theorem. Our proof requires very detailed information about spectrum of the matrix M; in Appendix we use standard methods of calculus to describe this spectrum.

2 Balance property and discrepancy

This section describes the main idea that will be later applied to find for any letter $a \in \{0, \ldots, m-1\}$ upper bound on the letter balance constant

$$c_a := \max\{|w|_a - |v|_a \,:\, v, w \in \mathcal{L}(\mathbf{u}) \text{ and } |w| = |v|\}\,.$$

The derivation of these bounds uses the following two ingredients.

- the m-bonacci sequence defined recursively

$$T_0 = T_1 = \ldots = T_{m-2} = 0, \qquad T_{m-1} = 1$$

  and

$$T_n = T_{n-1} + T_{n-2} + \ldots + T_{n-m} \tag{2}$$

  for any $n \geq m$;

- zeros $\beta \equiv \beta_0 > 1, \beta_1, \ldots, \beta_{m-1}$ of the polynomial

$$p(x) = x^m - x^{m-1} - \ldots - x - 1.$$

It is well known that $p(x)$ is an irreducible polynomial, its root β belongs to the interval $(1, 2)$, and the other roots (conjugates of β) are all of modulus less than 1. From now on, we order the roots $\beta_1, \ldots, \beta_{m-1}$ according to their arguments, i.e.,

$$0 \leq \arg(\beta_1) \leq \arg(\beta_2) \leq \cdots \leq \arg(\beta_{m-1}) < 2\pi\,. \tag{3}$$

The m-bonacci word is a fixed point of a primitive substitution. Therefore, density μ_a of any letter $a \in \mathcal{A}$ is well defined and positive, i.e.,

$$\mu_a = \lim_{n\to+\infty} \frac{|\mathbf{u}[n]|_a}{n} > 0,$$

where $\mathbf{u}[n]$ the prefix of $\mathbf{u}$ of length n. We refer to [9], where the problem of letter densities is studied in detail.

The value μ_a can be interpreted in the way that the "expected" number of letters a in the prefix $\mathbf{u}[n]$ is $\mu_a n$. A simple consequence of the definition of μ_a is the following observation.

Observation 1. *For any $\varepsilon > 0$ and for any positive integer N, there exist factors v and w in $\mathcal{L}(\mathbf{u})$ such that*

$$|v| = |w| = N, \qquad |w|_a \geq \mu_a N - \varepsilon \qquad \textit{and} \qquad |v|_a \leq \mu_a N + \varepsilon.$$

Proof. Assume that there exist $\varepsilon > 0$ and $N \geq 1$ such that for any factor w of length N, the inequality $|w|_a < \mu_a N - \varepsilon$ holds. It means that for the prefix of $\mathbf{u}$ of length $n = kN$, we obtain $|\mathbf{u}[n]| = |\mathbf{u}[kN]|_a < (\mu_a N - \varepsilon)k$. This implies $\mu_a = \lim_{n\to+\infty} \frac{|\mathbf{u}[n]|_a}{n} = \lim_{k\to+\infty} \frac{|\mathbf{u}[kN]|_a}{kN} < \mu_a - \frac{\varepsilon}{N}$, which is a contradiction. The proof of existence of v is analogous. □

The difference between the expected and actual number of letters a defines the discrepancy function $D_a : \mathbb{N} \to \mathbb{R}$;

$$D_a(n) = |\mathbf{u}[n]|_a - \mu_a n$$

for any $n \in \mathbb{N}$.

Lemma 2.1. *For any letter a, denote*

$$\Delta_a := \sup_{n\in\mathbb{N}} D_a(n) - \inf_{n\in\mathbb{N}} D_a(n).$$

Then $\Delta_a \leq c_a \leq 2\Delta_a$.

Proof. Let $w, v \in \mathcal{L}(\mathbf{u})$ be factors of the same length such that $c_a = |w|_a - |v|_a$. We can find prefixes W and V of $\mathbf{u}$ such that Ww and Vv are prefixes of $\mathbf{u}$ as well. Obviously

$$\begin{aligned} |w|_a - |v|_a &= |Ww|_a - |W|_a - |Vv|_a + |V|_a = D_a(|Ww|) - D_a(|W|) - D_a(|Vv|) + D_a(|V|) \\ &\leq 2 \sup_{n\in\mathbb{N}} D_a(n) - 2 \inf_{n\in\mathbb{N}} D_a(n) = 2\Delta_a. \end{aligned}$$

To deduce the lower bound on c_a, let us choose $\varepsilon > 0$. There exist prefixes of $\mathbf{u}$, say $\mathbf{u}[n_1]$ and $\mathbf{u}[n_2]$, such that $D_a(n_1) > \sup_{n\in\mathbb{N}} D_a(n) - \varepsilon$ and $D_a(n_2) < \inf_{n\in\mathbb{N}} D_a(n) + \varepsilon$, or equivalently

$$|\mathbf{u}[n_1]|_a > \mu_a n_1 + \sup_{n\in\mathbb{N}} D_a(n) - \varepsilon,$$
$$|\mathbf{u}[n_2]|_a < \mu_a n_2 + \inf_{n\in\mathbb{N}} D_a(n) + \varepsilon.$$

First suppose that $n_1 > n_2$ and put $N := n_1 - n_2$. Denote the suffix of $\mathbf{u}[n_1]$ of length N by $\tilde{W}$. Then $\tilde{W}$ contains at least $\mu_a N + \sup_{n\in\mathbb{N}} D_a(n) - \inf_{n\in\mathbb{N}} D_a(n) - 2\varepsilon$ letters a.

According to Observation 1, there exists a factor W of length N such that $|W|_a \le \mu_a N + \varepsilon$. Hence $c_a \ge |\tilde{W}|_a - |W|_a \ge \sup_{n\in\mathbb{N}} D_a(n) - \inf_{n\in\mathbb{N}} D_a(n) - 3\varepsilon = \Delta_a - 3\varepsilon$.

The case $n_1 < n_2$ is analogous. □

To find the value Δ_a, we apply the method of Adamczewski used in [1, 2]. Let us first recall the notation used in this method.

Let M be a matrix of the substitution (1). Since entries of M are defined as $M_{a,b} = |\varphi(b)|_a$ for $a, b \in \{0, 1, \ldots, m-1\}$, we have

$$M = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \in \mathbb{R}^{m\times m}.$$

By $\Psi(w)$ we denote the Parikh vector of the word $w \in \mathcal{A}^*$, i.e., $\Psi(w) = (|w|_0, |w|_1, \ldots, |w|_{m-1})^\intercal$. The matrix of a substitution helps effectively calculate the Parikh vector of an image w under φ. It is easy to see that

$$\Psi(\varphi(w)) = M\Psi(w) \qquad \text{for any } w \in \mathcal{A}^*. \tag{4}$$

Lemma 2.2. *For any prefix $\mathbf{u}[n]$ of the m-bonacci word $\mathbf{u}$, there exist $\ell \in \mathbb{N}$ and $\delta_0, \delta_1, \ldots, \delta_\ell \in \{0, 1\}$ such that*

$$\Psi(\mathbf{u}[n]) = \sum_{k=0}^{\ell} \delta_k M^k \Psi(0). \tag{5}$$

Moreover, for any choice of $\ell \in \{0, 1, 2, \ldots\}$ and $\delta_0, \ldots, \delta_\ell \in \{0, 1\}$, there exists a prefix $\mathbf{u}[n]$ of $\mathbf{u}$ such that (5) *holds.*

Proof. According to result [6], for any prefix there exist words $E_\ell \ne \epsilon, E_{\ell-1}, \ldots, E_1, E_0$ (ϵ is the empty word) such that

$$\mathbf{u}[n] = \varphi^\ell(E_\ell)\varphi^{\ell-1}(E_{\ell-1}) \cdots \varphi(E_1)E_0 \tag{6}$$

and for any k, the word E_k is a proper prefix of $\varphi(a)$ for some letter $a \in \mathcal{A}$.

For our substitution φ, the only proper prefixes of $\varphi(a)$ are $E_k = \epsilon$ and $E_k = 0$. Since the Parikh vector of a concatenation of words is the sum of their Parikh vectors, we have

$$\Psi(\mathbf{u}[n]) = \sum_{k=0}^{\ell} \delta_k \Psi\left(\varphi^k(0)\right),$$

where $\delta_k = 1$ if $E_k = 0$ and $\delta_k = 0$ if $E_k = \epsilon$. Applying formula (4) to $\Psi(\varphi^k(0))$, we get (5).

In general, not all sequences of $E_\ell, E_{\ell-1}, \ldots, E_1, E_0$ correspond to a prefix of $\mathbf{u}$. The relevant sequences are described by paths in so called prefix graph of substitution. Nevertheless, since for our substitution the equality $\varphi^m(0) = \varphi^{m-1}(0)\varphi^{m-2}(0)\cdots\varphi(0)0$ holds, any choice of $E_i \in \{\epsilon, 0\}$ gives a prefix of $\mathbf{u}$. □

Knowledge of the Parikh vector $\Psi(\mathbf{u}[n])$ enables us to compute discrepancy $D_a(n)$. To make arithmetic manipulation more elegant, Adamczewski denotes row vectors

$$\begin{aligned} h^{(0)} &= (1,0,\ldots,0) - \mu_0(1,1,\ldots,1), \\ h^{(1)} &= (0,1,\ldots,0) - \mu_1(1,1,\ldots,1), \\ &\vdots \\ h^{(m-1)} &= (0,\ldots,0,1) - \mu_{m-1}(1,1,\ldots,1), \end{aligned}$$

and expresses the discrepancy as the scalar product

$$D_a(n) = h^{(a)}\Psi(\mathbf{u}[n]). \tag{7}$$

Verification of the formula is straightforward.

Now we can formulate the main tool for estimation of c_a.

Proposition 2.3. *For any $a \in \{0,1,\ldots,m-1\}$ and $k \in \mathbb{N}$, denote*

$$g(a,k) = \left|\varphi^k(0)\right|_a - \mu_a \cdot \left|\varphi^k(0)\right| , \tag{8}$$

where μ_a is the density of the letter a in $\mathbf{u}$. Then

$$g(a,k) = T_{k+m-a-1} - \frac{1}{\beta^{a+1}} T_{k+m} \tag{9}$$

and

$$\sup_{n\in\mathbb{N}} D_a(n) - \inf_{n\in\mathbb{N}} D_a(n) = \sum_{k=0}^{+\infty} |g(a,k)|$$

Proof. At first, since $g(a,k)$ is nothing but $D_a(|\varphi^k(0)|)$, equation (7) gives $g(a,k) = h^{(a)}\Psi(\varphi^k(0))$. Using equation (4), we obtain $\Psi(\varphi^k(0)) = M^k\Psi(0)$, hence

$$g(a,k) = h^{(a)} M^k \Psi(0)\,. \tag{10}$$

This expression combined with equations (5) and (7) gives $D_a(n) = \sum_{k=0}^{\ell} \delta_k g(a,k)$, where $\delta_k \in \{0,1\}$. Clearly, $\sup\limits_{n\in\mathbb{N}} D_a(n) \leq \sum\limits_{\substack{k=0\\ g(a,k)>0}}^{+\infty} g(a,k)$ and $\inf\limits_{n\in\mathbb{N}} D_a(n) \geq \sum\limits_{\substack{k=0\\ g(a,k)<0}}^{+\infty} g(a,k)$. According to Lemma 2.2, any choice of δ_i's corresponds to a prefix of $\mathbf{u}[n]$, and, therefore, the equalities are reached in the previous inequalities. To sum up,

$$\sup_{n\in\mathbb{N}} D_a(n) - \inf_{n\in\mathbb{N}} D_a(n) = \sum_{\substack{k=0\\ g(a,k)>0}}^{+\infty} g(a,k) - \sum_{\substack{k=0\\ g(a,k)<0}}^{+\infty} g(a,k) = \sum_{k=0}^{+\infty} |g(a,k)|.$$

In order to prove equation (9), let us observe that

$$\begin{pmatrix} T_n \\ T_{n-1} \\ \vdots \\ T_{n-m+1} \end{pmatrix} = M \begin{pmatrix} T_{n-1} \\ T_{n-2} \\ \vdots \\ T_{n-m} \end{pmatrix}.$$

Since $(T_{m-1}, T_{m-2}, \ldots, T_0) = (1,0,0,\ldots,0) = (\Psi(0))^{\mathsf{T}}$, we get using (10)

$$g(a,k) = h^{(a)} M^k \Psi(0) = h^{(a)} \left(T_{m+k-1}, T_{m+k-2}, \ldots, T_k\right)^{\mathsf{T}}\,. \tag{11}$$

It is readily seen that the vector $\vec{\mu} = \left(\beta^{-1}, \beta^{-2}, \dots, \beta^{-m}\right)^{\mathsf{T}}$ is an eigenvector of M corresponding to the dominant eigenvalue β. Moreover, sum of components of $\vec{\mu}$ equals 1. It is well known that a vector $\vec{\mu}$ with these properties is the vector of letter densities, see [9]. It means that for any letter $a \in \{0, 1, \dots, m-1\}$, the density of letter a is $\mu_a = \beta^{-1-a}$. If we apply this fact to (11) and use the relation (2), we find

$$g(a,k) = T_{m+k-a-1} - \beta^{-a-1} T_{m+k}\,.$$

$\square$

Corollary 2.4. *The balance constants of the m-bonacci word satisfy*

$$c_a \le 2 \sum_{k=0}^{+\infty} |g(a,k)| \tag{12}$$

for all $a \in \mathcal{A}$.

Proof. The estimate follows easily from Lemma 2.1 and Proposition 2.3;

$$c_a \le 2\Delta_a = 2\left(\sup_{n\in\mathbb{N}} D_a(n) - \inf_{n\in\mathbb{N}} D_a(n)\right) = 2\sum_{k=0}^{+\infty} |g(a,k)|\,.$$

$\square$

Remark 1. *To estimate the sum $\sum_{k=0}^{+\infty} |g(a,k)|$, we will use the explicit formula for elements T_n of the m-bonacci sequence. The characteristic equation of* (9) *is the polynomial $p(x)$ with zeros $\beta = \beta_0, \beta_1, \dots, \beta_{m-1}$. Hence there exist constants $a_0, a_1, \dots, a_{m-1} \in \mathbb{C}$ such that*

$$T_n = a_0\beta_0^n + a_1\beta_1^n + \ldots + a_{m-1}\beta_{m-1}^n.$$

The constants $a_0, a_1, \dots, a_{m-1}$ depend on the initial values $T_0, T_1, \dots, T_{m-1}$ only. A standard calculation provides $T_n = \sum_{j=0}^{m-1} \frac{1}{p'(\beta_j)}\beta_j^n$, where p' denotes the derivative of the characteristic polynomial p.

Using (9), *we can conclude with*

$$g(a,k) = \sum_{j=1}^{m-1} \left(\frac{1}{\beta_j^{a+1}} - \frac{1}{\beta^{a+1}}\right) \frac{1}{p'(\beta_j)} \beta_j^{k+m}. \tag{13}$$

3 Numerical upper bounds on balance constant

According to Corollary 2.4, the letter balance constants of the m-bonacci word $\mathbf{u}$ can be estimated by the formula

$$c_a \le \left\lfloor 2\sum_{k=0}^{+\infty} |g(a,k)| \right\rfloor$$

for any letter $a \in \{0, 1, \dots, m-1\}$ and for all $m \ge 2$.

In this section we estimate the expressions $\left\lfloor 2\sum_{k=0}^{+\infty} |g(a,k)| \right\rfloor$ using a computer calculation. The calculations are very time-consuming for m above 10, therefore, we confine ourselves to $m \le 12$.

The calculation is based on the following strategy. We sum up the first n members of $(|g_{(a,k)}|)_{k=0}^{+\infty}$ and estimate the rest of them;

$$\sum_{k=0}^{+\infty} |g(a,k)| \le \sum_{k=0}^{n-1} |g(a,k)| + E, \qquad \text{where } E \text{ satisfies} \qquad E \ge \sum_{k=n}^{+\infty} |g(a,k)|.$$

Table 1: **4-bonacci** – $g(a,k)$ with quadruple of integer coefficients in linear combination of $g(a,0),\dots,g(a,3)$ and its signum.

	IC of $(g(a,k))_{k=0}^{3}$	$a=0$	$a=1$	$a=2$	$a=3$
$g(a,0)$	$(1,0,0,0)$	$+$	$-$	$-$	$-$
$g(a,1)$	$(0,1,0,0)$	$-$	$+$	$-$	$-$
$g(a,2)$	$(0,0,1,0)$	$-$	$-$	$+$	$-$
$g(a,3)$	$(0,0,0,1)$	$-$	$-$	$-$	$+$
$g(a,4)$	$(1,1,1,1)$	$+$	$-$	$-$	$-$
$g(a,5)$	$(1,2,2,2)$	$-$	$+$	$-$	$-$
$g(a,6)$	$(2,3,4,4)$	$-$	$-$	$+$	$-$
$g(a,7)$	$(4,6,7,8)$	$-$	$-$	$-$	$+$
$g(a,8)$	$(8,12,14,15)$	$+$	$+$	$-$	$-$
$g(a,9)$	$(15,23,27,29)$	$-$	$+$	$+$	$-$
$g(a,10)$	$(29,44,52,56)$	$-$	$-$	$+$	$+$
$g(a,11)$	$(56,85,100,108)$	$+$	$-$	$-$	$+$
$g(a,12)$	$(108,164,193,208)$	$+$	$+$	$-$	$-$

Formula (13) provides setting

$$E_{a,n} := |\beta_2|^n \sum_{j=1}^{m-1} \left| \left(\frac{1}{\beta_j^{a+1}} - \frac{1}{\beta^{a+1}} \right) \frac{1}{p'(\beta_j)} \right| \frac{|\beta_j|^n}{1-|\beta_j|}.$$

To conclude, we have to find an n big enough to satisfy

$$\left\lfloor 2 \sum_{k=0}^{n-1} |g(a,k)| \right\rfloor = \left\lfloor 2 \left(\sum_{k=0}^{n-1} |g(a,k)| + E_{a,n} \right) \right\rfloor . \tag{14}$$

Since we always compute on machines working in a finite precision, it is desirable to reduce the work with non-integer numbers. Therefore, we make use of the fact that, for a fixed letter a and the alphabet cardinality m, the sequence of numbers $g(a,k)$ satisfies the m-bonacci recurrence relation

$$g(a,n+m) = g(a,n+m-1) + \ldots + g(a,n)\,,$$

which follows from Proposition 2.3.

Let us demonstrate the method on the 4-bonacci word. The first step is calculating[1] $\operatorname{sgn} g(a,k)$ from (9) for all $k \in \{0,\dots,m-1\}$ (illustrated in Table 1). Then we express $\sum_{k=0}^{n-1} |g(a,k)|$ as an integer combination (IC) of $\begin{pmatrix} g(a,0) \\ \vdots \\ g(a,m-1) \end{pmatrix}$, which can be rewritten in the form $p + \frac{q}{\beta^{a+1}}$ for some $p,q \in \mathbb{Z}$ (this follows from Proposition 2.3) and then evaluated[1] (see Table 2). The final step is verification of the equality (14).

To make our procedure reliable with respect to possible rounding errors, we replace the estimated error $E_{a,m}$ by a constant $E > E_{a,m}$. If (14) holds, it is equal to the desired upper bound of c_a (but it may not be optimal). In the opposite case, we must increase n and repeat the procedure.

Our results obtained for $m \in \{2,\dots,12\}$ are summarized in Table 3.

To find lower bounds on the constant c, one needs to find two factors v,w of the m-bonacci word that are of the same length with $|w|_a - |v|_a$ big enough. Computer searching in the set of all factors is very time-consuming. Nevertheless, for any given $m \geq 4$ and any $a \in \{1,\dots,m-1\}$, a modification of the abelian co-decomposition method [12] allowed us to find a pair of factors v,w of the m-bonacci word such that $|v| = |w|$ and $|v|_a - |w|_a = 3$. For instance, if $m=4$, the words

$$v = 1\varphi^{12}(0)\varphi^{9}(0)\varphi^{5}(0)\varphi^{2}(0)\,,$$

[1] The calculation must be performed in an environment working in enough precision, e.g., Wolfram Mathematica.

Table 2: 4-**bonacci** – Estimates of $\sum_{k=0}^{+\infty} |g(a,k)|$ and the resulting upper bound on c_a.

	$a=0$	$a=1$	$a=2$	$a=3$
$\sum_{k=0}^{12} \lvert g(a,k)\rvert$ as IC	$\begin{pmatrix}123\\183\\215\\232\end{pmatrix}$	$\begin{pmatrix}39\\63\\71\\76\end{pmatrix}$	$\begin{pmatrix}-133\\-201\\-233\\-254\end{pmatrix}$	$\begin{pmatrix}-47\\-71\\-83\\-86\end{pmatrix}$
$\sum_{k=0}^{12} \lvert g(a,k)\rvert$ symbolic	$1664-\frac{3205}{\beta}$	$286-\frac{1057}{\beta^2}$	$\frac{3499}{\beta^3}-487$	$\frac{1209}{\beta^4}-86$
$\sum_{k=0}^{12} \lvert g(a,k)\rvert$ numerical	1.2778	1.5157	1.5611	1.5776
$E_{a,13}$	0.20054	0.22213	0.25916	0.31056
$\sum_{k=0}^{12} \lvert g(a,k)\rvert + E$	1.49844	1.76006	1.84618	1.91919
c_a upper bound	2	3	3	3

Table 3: Upper estimates of c_a for $m \in \{2,\ldots,12\}$, $a \in \{0,\ldots,m-1\}$.

$m \setminus a$	0	1	2	3	4	5	6	7	8	9	10	11
2	1	1	×	×	×	×	×	×	×	×	×	×
3	2	2	2	×	×	×	×	×	×	×	×	×
4	2	3	3	3	×	×	×	×	×	×	×	×
5	2	3	3	3	3	×	×	×	×	×	×	×
6	3	3	4	4	4	4	×	×	×	×	×	×
7	3	4	4	4	4	4	4	×	×	×	×	×
8	3	4	4	4	4	4	4	4	×	×	×	×
9	3	4	5	5	5	5	5	5	5	×	×	×
10	3	5	5	5	5	5	5	5	5	5	×	×
11	4	5	5	6	6	6	6	6	6	6	6	×
12	4	5	6	6	6	6	6	6	6	6	6	6

$$w = \left(\varphi^9(0)\varphi^8(0)\varphi^5(0)\varphi^2(0)\right)^{-1}\varphi^{11}(00)\varphi^{10}(0)\varphi^7(0)\varphi^6(0)\varphi^4(0)\varphi^3(0)\varphi^2(0)0$$

are factors of $\mathbf{u}$ such that $|v| = |w| = 3305$, $|v|_1 - |w|_1 = 3$. Similarly, if $m = 5$, the words

$$v = 1\varphi^{14}(0)\varphi^{11}(0)\varphi^6(0)\varphi^2(0)\,,$$

$$w = \left(\varphi^{11}(0)\varphi^{10}(0)\varphi^6(0)\varphi^2(0)\right)^{-1}\varphi^{13}(00)\varphi^{12}(0)\varphi^9(0)\varphi^8(0)\varphi^7(0)\varphi^5(0)\varphi^3(0)\varphi^2(0)0$$

are factors of $\mathbf{u}$ such that $|v| = |w| = 15481$, $|v|_1 - |w|_1 = 3$.

Therefore, we can conclude with the following theorem.

Theorem 3.1. *For $m \in \{4,5\}$, the m-bonacci word is c-balanced with $c = 3$ and this bound cannot be improved.*

For $m \in \{6,\dots,12\}$, the m-bonacci word is c-balanced for $c = \lceil \frac{m+1}{2} \rceil$.

4 Balance property of letters in the m-bonacci word

The numerical calculation, performed in Section 3, is convenient only for small values of m. In the rest of the paper we develop a technique to estimate the constant c for the balance property of the m-bonacci word for a general m. The calculation will be again based on formula (12), but this time we bring in an improvement. Instead of estimating the sums $\sum_{k=0}^{+\infty}|g(a,k)|$ for all letters $a \in \mathcal{A}$, we show that in case of the m-bonacci word, the balance constants c_a for $a = 1,2,\dots,m-1$ can be estimated by a simple formula in terms of c_0 providing that c_0 is small enough, see the following observation.

Proposition 4.1. *Let $m \geq 4$. If $c_0 \leq 2^{m-1} - 3$, then*

$$c_j \leq \left(2 - \frac{1}{2^j}\right)c_0 + 4\left(1 - \frac{1}{2^j}\right) \tag{15}$$

for each $j = 1,2,\dots,m-1$. In particular, the m-bonacci word is c-balanced with $c = 2c_0 + 3$.

With regard to this proposition, it will be sufficient to estimate $\sum_{k=0}^{+\infty}|g(a,k)|$ and use formula (12) just once, for $a = 0$. All the remaining constants c_a for $a = 1,\dots,m-1$ can be then easily estimated using formula (15).

Before we prove Proposition 4.1, we derive two simple observations.

Observation 2. *For any factor f of $\mathbf{u}$ and for each $j \in \{1,\dots,m-1\}$, it holds*

$$|f|_0 = |\varphi^j(f)|_j \qquad \text{and} \qquad |f| = |\varphi^j(f)|_{j-1}.$$

Proof. From the form of the substitution (1), we see $|w|_{j-1} = |\varphi(w)|_j$ and $|w| = |\varphi(w)|_0$ for any factor w and letter $j = 1,2,\dots,m-1$. Applying these relations on $w = f$, $w = \varphi(f),\dots,$ $w = \varphi^{j-1}(f)$, we get the formulae in the observation. ☐

Observation 3. *If f is a factor of $\mathbf{u}$ such that $|f| \leq 2^m$, then $|f|_0 \leq \frac{1}{2}|f| + 1$.*

Proof. The form of the substitution φ implies that 00 is the longest block of zeros occurring in $\mathbf{u}$. Further, with exception of this block, the letter 0 is always sandwiched by nonzero letters. It is easy to see that the shortest factor $w \neq 00$, with the prefix 00 and the suffix 00 such that w has no other occurrences of 00, is the factor $w = 0\varphi^m(0)0$. Since $|w| = 2^m + 1$, any factor f with $|f| \leq 2^m$ contains at most one block 00. This implies the inequality for $|f|_0$ stated in the observation. ☐

The following lemma is the combinatorial core for the proof of Proposition 4.1.

Lemma 4.2. *Let $j \in \{1,\dots,m-1\}$. If $c_{j-1} \leq 2^m - 2$, then*

$$c_j \leq c_0 + 2 + \frac{c_{j-1}}{2}\,. \tag{16}$$

Proof. With respect to the definition of c_j, there exists a pair of factors v and w such that

$$|v| = |w| \qquad \text{and} \qquad |v|_j - |w|_j = c_j. \tag{17}$$

Without loss of generality, we can assume that v and w is the shortest possible pair satisfying (17). Then v and w are in the form $v = j \cdots j$ and $w = \ell \cdots \ell'$ for certain $\ell, \ell' \neq j$. Moreover, we can assume that jw is a factor of $\mathbf{u}$ (otherwise we replace $w = \mathbf{u}_i \cdots \mathbf{u}_{i+|w|-1}$ by $w' = \mathbf{u}_{i-i'} \cdots \mathbf{u}_{i+|w|-1-i'}$ without violating equations (17)).

Because of the form of v, there exists a factor $V = 0V' \in \mathcal{L}(\mathbf{u})$ such that $v = j\varphi^j(V')$. Clearly, v is a suffix of $\varphi^j(0V') = \varphi^j(V)$.

Let wzj be a factor of $\mathbf{u}$ such that $|z|_j = 0$ (we extend the factor w to the right up to the next letter j). As $jwzj \in \mathcal{L}(\mathbf{u})$ by assumption, there exists a factor W such that $wzj = \varphi^j(W0)$.

Observation 2 implies

- $|V|_0 = 1 + |V'|_0 = 1 + |\varphi^j(V')|_j = |j\varphi^j(V')|_j = |v|_j$
- $|W|_0 = |W0|_0 - 1 = |\varphi^j(W0)|_j - 1 = |wzj|_j - 1 = |w|_j$
- $|V| = 1 + |V'| = 1 + |\varphi^j(V')|_{j-1} = 1 + |v|_{j-1}$
- $|W| = |W0| - 1 = |\varphi^j(W0)|_{j-1} - 1 = |wzj|_{j-1} - 1$

Together, we have deduced

$$|V|_0 - |W|_0 = c_j \qquad \text{and} \qquad |V| - |W| = |v|_{j-1} - |w|_{j-1} - |z|_{j-1} + 2. \tag{18}$$

We distinguish two cases:

- *Case* $|V| \leq |W|$. Let $\hat{V} = Vx$ be a factor of $\mathbf{u}$ such that $|\hat{V}| = |W|$. From the definition of c_0 and (18) we get $c_0 \geq |\hat{V}|_0 - |W|_0 \geq |V|_0 - |W|_0 = c_j$. Thus $c_j \leq c_0 + 2 + \frac{c_{j-1}}{2}$ holds trivially.
- *Case* $|V| > |W|$. Let $\hat{W} = Wy$ be a factor of $\mathbf{u}$ such that $|\hat{W}| = |V|$. Then $c_0 \geq |V|_0 - |\hat{W}|_0 = |V|_0 - |W|_0 - |y|_0 = c_j - |y|_0$ due to (18). To bound length of y, we apply Equation (18). It gives $|y| = |V| - |W| = |v|_{j-1} - |w|_{j-1} - |z|_{j-1} + 2 \leq |v|_{j-1} - |w|_{j-1} + 2 \leq c_{j-1} + 2$. With regard to the assumption $c_{j-1} \leq 2^m - 2$, we have $|y| \leq 2^m$. Therefore, $|y|_0 \leq \frac{1}{2}|y| + 1$ due to Observation 3. To sum up, $c_0 \geq c_j - \left(\frac{1}{2}(c_{j-1} + 2) + 1\right) \geq c_j - 2 - \frac{1}{2}c_{j-1}$.

□

Proof of Proposition 4.1. Let us assume that $c_0 \leq 2^{m-1} - 3$. We prove equation (15) by induction on j.

I. Let $j = 1$. It holds $c_0 \leq 2^{m-1} - 3 \leq 2^m - 2$ by assumption, therefore, we can use Lemma (4.2). It implies $c_1 \leq c_0 + 2 + \frac{c_0}{2}$, hence indeed $c_j \leq \left(2 - \frac{1}{2^1}\right) c_0 + 4\left(1 - \frac{1}{2^1}\right)$.

II. Let $j > 1$ *and equation* (15) *hold for* $j - 1$. Inequality $c_0 \leq 2^{m-1} - 3$ implies

$$c_{j-1} \leq \left(2 - \frac{1}{2^{j-1}}\right) c_0 + 4\left(1 - \frac{1}{2^{j-1}}\right) < 2c_0 + 4 \leq 2(2^{m-1} - 3) + 4 = 2^m - 2\,.$$

It allows us to apply Lemma (4.2). Equation (16) gives

$$c_j \leq c_0 + 2 + \frac{1}{2}c_{j-1} \leq c_0 + 2 + \frac{1}{2}\left(\left(2 - \frac{1}{2^{j-1}}\right) c_0 + 4\left(1 - \frac{1}{2^{j-1}}\right)\right) = \left(2 - \frac{1}{2^j}\right) c_0 + 4\left(1 - \frac{1}{2^j}\right).$$

In particular, (15) yields $c_j < 2c_0 + 4$. As $c = \max\{c_j : j = 0, 1, \ldots, m-1\}$ and c and c_0 are integers, necessarily $c \leq 2c_0 + 3$. □

5 Estimate of $\sum_{k=0}^{+\infty} |g(0,k)|$

As anticipated in Section 4, the balance constant c_0 will be obtained using formula 12. Therefore, we need to estimate the sum $\sum_{k=0}^{+\infty} |g(0,k)|$. This is the topic of this section; since we deal with the letter $a = 0$ only, we abbreviate the symbol $g(0,k)$ to $g(k)$.

The sum $\sum_{k=0}^{+\infty} |g(0,k)|$ will be estimated by splitting it into two parts, $\sum_{k=0}^{2m-1} |g(k)|$ and $\sum_{k=2m}^{+\infty} |g(k)|$, and estimating each of them separately. In Sections 5.1 and 5.2 we show that

$$\sum_{k=0}^{2m-1} |g(k)| < \frac{5}{4} \qquad \text{and} \qquad \sum_{k=2m}^{+\infty} |g(k)| < \frac{A}{2\pi} m + 1 \quad \text{for all } m \geq 4\,.$$

To get these estimates we will exploit bounds on absolute values and arguments of zeros of polynomials $p(x)$, derived in Appendix A.

5.1 An upper bound on the sum $\sum\limits_{k=0}^{2m-1} |g(k)|$

At first we express $g(k)$'s for all $k = 0, 1, \ldots, 2m-1$ and determine their signs. Recall that $\mu_0 = 1/\beta$, therefore, due to equation (8), it holds

$$g(k) = \left|\varphi^k(0)\right|_0 - \frac{1}{\beta} \cdot \left|\varphi^k(0)\right| \,. \tag{19}$$

In the sequel we use the following formula to calculate $g(k)$ for all $k \leq 2m-1$.

Proposition 5.1. *It holds*

$$|\varphi^k(0)| = \begin{cases} 2^k & \textit{for } k = 0, \ldots, m-1\,; \\ 2^k - 2^{k-m} - (k-m)2^{k-m-1} & \textit{for } k = m, \ldots, 2m-1\,. \end{cases} \tag{20}$$

Proof. The identity $\varphi^k(0) = \varphi\left(\varphi^{k-1}(0)\right)$ together with the substitution (1) implies

$$|\varphi^k(0)| = 2|\varphi^{k-1}(0)| - |\varphi^{k-1}(0)|_{m-1}\,. \tag{21}$$

Let us distinguish two cases.

- *Case* $k \leq m-1$. It holds $\varphi^0(0) = 0$ and $|\varphi^k(0)|_{m-1} = 0$ for all $k \leq m-2$, hence $|\varphi^k(0)| = 2|\varphi^{k-1}(0)|$ for all $k \leq m-1$.
- *Case* $k \geq m$. We prove equation (20) for $k \in \{m, m+1, \ldots, 2m-1\}$ by induction on k.

I. $k = m$. We have $|\varphi^{m-1}(0)|_{m-1} = 1$, hence $|\varphi^m(0)| = 2|\varphi^{m-1}(0)| - 1 = 2^m - 1$. Since $2^m - 1 = 2^m - 2^{m-m} - (m-m)2^{m-m-1}$, the statement holds true for $k = m$.

II. $k \geq m+1$. Let $|\varphi^{k-1}(0)| = 2^{k-1} - 2^{k-1-m} - (k-1-m)2^{k-1-m-1}$. The identity $|\varphi^{k-1}(0)|_{m-1} = |\varphi^{k-1-m}(0)|$, valid for every $k \geq m+1$, allows us to use the formula (21) in the form

$$|\varphi^k(0)| = 2|\varphi^{k-1}(0)| - |\varphi^{k-1-m}(0)|\,.$$

Since $k-1-m < m-1$, we can apply the results obtained above $k \leq m-1$, whence we get

$$|\varphi^k(0)| = 2\left(2^{k-1} - 2^{k-1-m} - (k-1-m)2^{k-1-m-1}\right) - 2^{k-1-m} = 2^k - 2^{k-m} - (k-m)2^{k-m-1}\,.$$

□

To determine signs of $g(k)$'s defined by (19), we need a fine estimate on β . Let us recall that β is the dominant eigenvalue of the matrix of substitution M and thus a zero of its characteristic polynomial $p(x) = x^m - x^{m-1} - x^{m-2} - \cdots - x - 1$.

Proposition 5.2. *It holds*

$$g(0) = 1 - \frac{1}{\beta} > 0\,;$$
$$g(k) = 2^{k-1}\left(1 - \frac{2}{\beta}\right) < 0 \quad \textit{for } k = 1, \ldots, m-1\,;$$
$$g(m) = 2^{m-1}\left(1 - \frac{2}{\beta}\right) + \frac{1}{\beta} > 0\,;$$
$$g(k) = \left(1 - \frac{2}{\beta}\right)\left(2^{k-1} - (k+1-m)2^{k-m-2}\right) + \frac{1}{\beta}\cdot 2^{k-m-1} < 0 \quad \textit{for } k = m+1, \ldots, 2m-1\,.$$

Proof. The formula for $g(0)$ follows immediately from equation (19).

For every $k \geq 1$, it holds $|\varphi^k(0)|_0 = |\varphi^{k-1}(0)|$, hence

$$g(k) = |\varphi^{k-1}(0)| - \frac{1}{\beta}\cdot|\varphi^k(0)|\,,$$

cf. equation (19). All the formulae for $g(k)$ listed in Proposition 5.2 then follow easily from equation (20).

In the rest of the proof we show that $g(0) > 0$, $g(m) > 0$, and $g(k) < 0$ for all $k \in \{1, \ldots, m-1\} \cup \{m+1, \ldots, 2m-1\}$.

At first, $\beta \in (1,2)$ immediately implies $g(0) > 0$ and $g(k) < 0$ for all $k \in \{1, \ldots, m-1\}$.

As for $k = m$, we shall show that

$$2^{m-1}\left(1 - \frac{2}{\beta}\right) + \frac{1}{\beta} > 0\,.$$

This inequality is equivalent to

$$2 - \beta < \frac{1}{2^{m-1}}\,,$$

which is valid due to (43) from Appendix, because $1/2^{m-1} > 1/(2^m - (m+1)/2)$ for all $m \geq 2$. Similarly, if $k \geq m+1$, we need to prove that

$$\left(1 - \frac{2}{\beta}\right)\left(2^{k-1} - (k+1-m)2^{k-m-2}\right) + \frac{1}{\beta}\cdot 2^{k-m-1} < 0 \quad \text{for } k = m+1, \ldots, 2m-1\,;$$

i.e.,

$$2 - \beta > \frac{1}{2^m - \frac{k+1-m}{2}} \qquad \text{for all } k = m+1, \ldots, 2m-1\,.$$

Since $k+1-m \leq m$, the validity immediately follows from inequalities (43). $\square$

Proposition 5.3. *It holds*

$$\sum_{k=0}^{2m-1} |g(k)| = 1 + \left(\frac{2}{\beta} - 1\right)\left[2^m\left(2^{m-1} - 1\right) - (m-1)2^{m-2}\right] - \frac{1}{\beta}\left(2^{m-1} - 1\right) < 1 + \frac{1}{4}\,. \tag{22}$$

Proof. Proposition 5.2 implies

$$\sum_{k=0}^{2m-1} |g(k)| = g(0) - \sum_{k=1}^{m-1} g(k) + g(m) - \sum_{k=m+1}^{2m-1} g(k)\,.$$

When we substitute for $g(k)$ from Proposition 5.2, we obtain

$$g(0) + g(m) = 1 + 2^{m-1}\left(1 - \frac{2}{\beta}\right)\,,$$

$$-\sum_{k=1}^{m-1} g(k) = -\sum_{k=1}^{m-1} 2^{k-1}\left(1-\frac{2}{\beta}\right) = -(2^{m-1}-1)\left(1-\frac{2}{\beta}\right),$$

and, in a similar way, we get

$$-\sum_{k=m+1}^{2m-1} g(k) = -\left(1-\frac{2}{\beta}\right)\left[2^m(2^{m-1}-1)-(m-1)2^{m-2}\right] - \frac{1}{\beta}\left(2^{m-1}-1\right).$$

Summing up these expressions, we get formula (22).

In the rest of the proof we show that $\sum_{k=0}^{2m-1}|g(k)| < 1+1/4$, which is obviously equivalent to

$$(2-\beta)\left[2^m\left(2^{m-1}-1\right)-(m-1)2^{m-2}\right] - 2^{m-1}+1 < \frac{\beta}{4},$$

and also to

$$(2-\beta)\left[2^m\left(2^{m-1}-1\right)-(m-1)2^{m-2}+\frac{1}{4}\right] - 2^{m-1}+1 < \frac{1}{2}.$$

Using inequality (43), we obtain

$$\begin{aligned}
&(2-\beta)\left[2^m\left(2^{m-1}-1\right)-(m-1)2^{m-2}+\frac{1}{4}\right] - 2^{m-1}+1 \\
&\qquad \le \frac{1}{2^m-\frac{m+1}{2}}\left[2^m\left(2^{m-1}-1\right)-(m-1)2^{m-2}+\frac{1}{4}\right] - 2^{m-1}+1 \\
&= \frac{2^{m-1}-1-\frac{m-1}{4}+\frac{1}{2^{m+2}}}{1-\frac{m+1}{2^{m+1}}} - 2^{m-1}+1 = \frac{-\frac{m-1}{4}+\frac{1}{2^{m+2}}+\frac{m+1}{4}-\frac{m+1}{2^{m+1}}}{1-\frac{m+1}{2^{m+1}}} = \frac{1}{2}\cdot\frac{1-\frac{2m+1}{2^{m+1}}}{1-\frac{m+1}{2^{m+1}}} < \frac{1}{2}.
\end{aligned}$$

□

5.2 An upper bound on the sum $\sum_{k=2m}^{+\infty}|g(k)|$

Proposition 5.4. *For any $k \in \mathbb{N}$ we have*

$$|g(k)| \le \frac{1}{2(m-1)}\sum_{j=1}^{m-1}|\Re(\beta_j)-1|\cdot|\beta_j|^k. \tag{23}$$

Proof. With regard to equation (42) from Appendix,

$$p'(x) = \frac{(m+1)x^m-2mx^{m-1}}{x-1} - \frac{x^{m+1}-2x^m+1}{(x-1)^2} = \frac{(m+1)x^m-2mx^{m-1}}{x-1} - \frac{p(x)}{x-1}.$$

Since $p(\beta_j)=0$ for every eigenvalue of M, we have

$$p'(\beta_j) = \frac{(m+1)\beta_j^m-2m\beta_j^{m-1}}{\beta_j-1} = \frac{(m+1)\beta_j-2m}{\beta_j-1}\beta_j^{m-1}.$$

Therefore, due to (13),

$$g(k) = \sum_{j=1}^{m-1}\left(\frac{1}{\beta_j}-\frac{1}{\beta}\right)\frac{\beta_j^{k+m}}{\frac{(m+1)\beta_j-2m}{\beta_j-1}\beta_j^{m-1}} = \sum_{j=1}^{m-1}\frac{\beta-\beta_j}{\beta}\cdot\frac{\beta_j-1}{(m+1)\beta_j-2m}\beta_j^k.$$

As $g(k)$ is real, we can write

$$g(k) = \sum_{j=1}^{m-1}\frac{1}{\beta}\Re\left(\frac{\beta-\beta_j}{(m+1)\beta_j-2m}(\beta_j-1)\beta_j^k\right), \tag{24}$$

and estimate

$$|g(k)| \le \sum_{j=1}^{m-1} \frac{1}{\beta} \left| \Re\left(\frac{\beta-\beta_j}{(m+1)\beta_j - 2m} (\beta_j - 1)\beta_j^k \right) \right|$$
$$\le \sum_{j=1}^{m-1} \frac{1}{\beta} \left| \frac{\beta-\beta_j}{(m+1)\beta_j - 2m} \right| \cdot |\Re(\beta_j) - 1| \cdot \left|\beta_j^k\right| .$$

To finish our proof we will deduce for all $j = 1, \dots, m-1$,

$$\frac{1}{\beta} \left| \frac{\beta-\beta_j}{(m+1)\beta_j - 2m} \right| \le \frac{1}{2(m-1)} . \tag{25}$$

Since

$$\frac{1}{\beta} \left| \frac{\beta-\beta_j}{(m+1)\beta_j - 2m} \right| = \frac{1}{2(m-1)} \left| \frac{m-1-(m-1)\frac{\beta_j}{\beta}}{m-(m+1)\frac{\beta_j}{2}} \right| , \tag{26}$$

it suffices to prove that

$$\left| \frac{m-1-(m-1)\frac{\beta_j}{\beta}}{m-(m+1)\frac{\beta_j}{2}} \right| \le 1 .$$

We have

$$\left| \frac{m-1-(m-1)\frac{\beta_j}{\beta}}{m-(m+1)\frac{\beta_j}{2}} \right|^2 = \frac{\left[m-1-\frac{m-1}{\beta}\Re(\beta_j)\right]^2 + \left[\frac{m-1}{\beta}\Im(\beta_j)\right]^2}{\left[m-\frac{m+1}{2}\Re(\beta_j)\right]^2 + \left[\frac{m+1}{2}\Im(\beta_j)\right]^2} . \tag{27}$$

Lemma A.1 implies $2-\beta < \frac{2}{m+1} < \frac{4}{m+1}$; hence

$$\frac{m-1}{\beta} < \frac{m+1}{2} . \tag{28}$$

Therefore

$$\left[\frac{m-1}{\beta}\Im(\beta_j)\right]^2 < \left[\frac{m+1}{2}\Im(\beta_j)\right]^2 . \tag{29}$$

In what follows we demonstrate that

$$\left| m-1-\frac{m-1}{\beta}\Re(\beta_j) \right| < \left| m-\frac{m+1}{2}\Re(\beta_j) \right| . \tag{30}$$

Since $\beta \in (1,2)$ and $|\beta_j| < 1$, we have

$$0 < m-1-\frac{m-1}{\beta}\Re(\beta_j) = m-\frac{m+1}{2}\Re(\beta_j) - 1 + \left(\frac{m+1}{2} - \frac{m-1}{\beta}\right)\Re(\beta_j) .$$

It holds $\Re(\beta_j) < 1$, and the expression $\frac{m+1}{2} - \frac{m-1}{\beta}$ is positive due to equation 28; therefore

$$-1 + \left(\frac{m+1}{2} - \frac{m-1}{\beta}\right)\Re(\beta_j) < -1 + \frac{m+1}{2} - \frac{m-1}{\beta} = -(m-1)\left(\frac{1}{\beta} - \frac{1}{2}\right) < 0 .$$

Hence

$$0 < m-1-\frac{m-1}{\beta}\Re(\beta_j) < m-\frac{m+1}{2}\Re(\beta_j) ,$$

i.e., (30) holds true. Equation (27) together with inequalities (29) and (30) implies

$$\left| \frac{m-1-(m-1)\frac{\beta_j}{\beta}}{m-(m+1)\frac{\beta_j}{2}} \right|^2 < \frac{[2m-(m+1)\Re(\beta_j)]^2 + [(m+1)\Im(\beta_j)]^2}{[2m-(m+1)\Re(\beta_j)]^2 + [(m+1)\Im(\beta_j)]^2} = 1 .$$

□

Corollary 5.5.

$$\sum_{k=2m}^{+\infty} |g(k)| \leq \frac{1}{2(m-1)} \sum_{j=1}^{m-1} \cdot \frac{|\Re(\beta_j)-1|}{1-|\beta_j|} \cdot \frac{1}{|2-\beta_j|^2}\,. \tag{31}$$

Proof. Using (23), we can estimate

$$\sum_{k=2m}^{+\infty} |g(k)| \leq \frac{1}{2(m-1)} \sum_{j=1}^{m-1} |\Re(\beta_j)-1| \sum_{k=2m}^{+\infty} |\beta_j^k| = \frac{1}{2(m-1)} \sum_{j=1}^{m-1} |\Re(\beta_j)-1| \frac{|\beta_k|^{2m}}{1-|\beta_j|}\,.$$

Finally, we use Observation 4 to rewrite $|\beta_j|^{2m} = 1/|2-\beta_j|^2$. ☐

At this stage we apply the information on $|\beta_j|$ for $j=1,\dots,m-1$, derived in Lemma A.2.

Proposition 5.6. *It holds*

$$\sum_{k=2m}^{+\infty} |g(k)| < \frac{1}{2(m-1)} \cdot \frac{1}{1-\frac{2\ln 3}{3m}} \left(\frac{2m}{1-\frac{\ln 3}{m}} \sum_{j=1}^{m-1} \frac{1-\cos\gamma_j}{(5-4\cos\gamma_j)\ln(5-4\cos\gamma_j)} + \sum_{j=1}^{m-1} \frac{\cos\gamma_j}{5-4\cos\gamma_j} \right)\,. \tag{32}$$

Proof. We will estimate summands from inequality (31). In the notation $\beta_j = B_j \mathrm{e}^{\mathrm{i}\gamma_j}$, we have

$$\frac{|\Re(\beta_j)-1|}{1-|\beta_j|} = \frac{1-B_j\cos\gamma_j}{1-B_j} = \frac{1-\cos\gamma_j}{1-B_j} + \cos\gamma_j\,,$$

thus equation (44) from Appendix implies

$$\frac{1-\cos\gamma_j}{1-B_j} + \cos\gamma_j \leq \frac{1-\cos\gamma_j}{\ln(5-4\cos\gamma_j)} \cdot \frac{2m}{1-\frac{\ln 3}{m}} + \cos\gamma_j\,. \tag{33}$$

Concerning the term $1/|2-\beta_j|^2$, it holds

$$\frac{1}{|2-\beta_j|^2} = \frac{1}{4-4B_j\cos\gamma_j+B_j^2} = \frac{1}{5-4\cos\gamma_j+4(1-B_j)\cos\gamma_j-2(1-B_j)+(1-B_j)^2}$$
$$< \frac{1}{5-4\cos\gamma_j} \cdot \frac{1}{1-(1-B_j)\frac{2-4\cos\gamma_j}{5-4\cos\gamma_j}}\,.$$

It is easy to see that $\frac{2-4\cos\gamma}{5-4\cos\gamma} \leq \frac{2}{3}$, therefore, it suffices to estimate $1-B_j$ from above. Since

$$B_j = \frac{1}{\sqrt[2m]{4-4B_j\cos\gamma_j+B_j^2}} > \frac{1}{\sqrt[2m]{9}} = \frac{1}{\sqrt[m]{3}}$$

and

$$\sqrt[m]{3} = \mathrm{e}^{\frac{\ln 3}{m}} < \left[\left(1+\frac{1}{\frac{m}{\ln 3}-1}\right)^{\frac{m}{\ln 3}} \right]^{\frac{\ln 3}{m}} = \frac{\frac{m}{\ln 3}}{\frac{m}{\ln 3}-1}\,,$$

it holds $B_j > \frac{\frac{m}{\ln 3}-1}{\frac{m}{\ln 3}}$. Hence $1-B_j < \frac{\ln 3}{m}$ for all $j=1,\dots,m-1$. Consequently,

$$\frac{1}{|2-\beta_j|^2} < \frac{1}{5-4\cos\gamma_j} \cdot \frac{1}{1-\frac{2}{3}\cdot\frac{\ln 3}{m}}\,. \tag{34}$$

Inequality (31) combined with estimates (33) and (34) leads to formula (32). ☐

The following lemma is an essential component of our calculation. It uses the information on γ_j obtained in Lemma A.3.

Lemma 5.7. *It holds*

$$\sum_{j=1}^{m-1} \frac{1-\cos\gamma_j}{(5-4\cos\gamma_j)\ln(5-4\cos\gamma_j)} \le \frac{m}{2\pi}\int_0^{2\pi} \frac{1-\cos x}{(5-4\cos\gamma_j)\ln(5-4\cos x)}\,\mathrm{d}x - \frac{1}{6} + \frac{m-1}{m}\cdot\frac{\pi}{16}\left(1+\frac{1}{36}\right). \tag{35}$$

Proof. Let us denote

$$f(x) = \frac{1-\cos x}{\ln(5-4\cos x)}\,;$$

then

$$\sum_{j=1}^{m-1} \frac{1-\cos\gamma_j}{\ln(5-4\cos\gamma_j)} = \frac{m}{2\pi}\sum_{j=1}^{m-1}\frac{2\pi}{m} f(\gamma_j) \tag{36}$$

The estimate (52) implies $\gamma_j \in \left(\frac{2\pi}{m}\left(j-\frac{1}{2}\right), \frac{2\pi}{m}\left(j+\frac{1}{2}\right)\right)$. Therefore, the sum (36) is a Riemann sum of the function f with respect to the tagged partition

$$\frac{\pi}{m} = x_0 < x_1 < \ldots < x_{m-1} = 2\pi - \frac{\pi}{m}\,, \qquad \text{where } x_j = \frac{2\pi}{m}\left(j+\frac{1}{2}\right),$$

of interval $\left[\frac{\pi}{m}, 2\pi-\frac{\pi}{m}\right]$. Let us rewrite the summands of (36) using a trivial identity

$$\frac{2\pi}{m} f(\gamma_j) = \int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x + \int_{x_{j-1}}^{x_j} (f(\gamma_j) - f(x))\,\mathrm{d}x\,.$$

Since

$$f(\gamma_j) - f(x) \le |x-\gamma_j|\cdot \max_{x\in(x_{j-1},x_j)}\{|f'(x)|\} \le |x-\gamma_j|\cdot \max_{x\in[0,2\pi)}\{|f'(x)|\}\,,$$

we have

$$\frac{2\pi}{m} f(\gamma_j) \le \int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x + \max_{x\in[0,2\pi)}\{|f'(x)|\}\int_{x_{j-1}}^{x_j} |x-\gamma_j|\,\mathrm{d}x\,.$$

Now we apply another identity, valid for any $\gamma_j \in [x_{j-1}, x_j]$,

$$\int_{x_{j-1}}^{x_j} |x-\gamma_j|\,\mathrm{d}x = \int_{x_{j-1}}^{\gamma_j} (\gamma_j - x)\,\mathrm{d}x + \int_{\gamma_j}^{x_j} (x-\gamma_j)\,\mathrm{d}x = \int_0^{\gamma_j - x_{j-1}} x\,\mathrm{d}x + \int_0^{x_j-\gamma_j} x\,\mathrm{d}x\,,$$

which provides us, using the estimate (52), the inequality

$$\int_{x_{j-1}}^{x_j} |x-\gamma_j|\,\mathrm{d}x \le \int_0^{\frac{\pi}{m}+\frac{\pi}{6m}} x\,\mathrm{d}x + \int_0^{\frac{\pi}{m}-\frac{\pi}{6m}} x\,\mathrm{d}x = \frac{\pi^2}{m^2}\left(1+\frac{1}{36}\right).$$

Hence

$$\frac{2\pi}{m} f(\gamma_j) \le \int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x + \max_{x\in[0,2\pi)}\{|f'(x)|\}\frac{\pi^2}{m^2}\left(1+\frac{1}{36}\right).$$

Consequently,

$$\sum_{j=1}^{m-1} f(\gamma_j) \quad \le \quad \frac{m}{2\pi}\left(\int_{\frac{\pi}{m}}^{2\pi-\frac{\pi}{m}} f(x)\,\mathrm{d}x + (m-1)\max_{x\in[0,2\pi)}\{|f'(x)|\}\frac{\pi^2}{m^2}\left(1+\frac{1}{36}\right)\right).$$

Furthermore, it can be checked that $f(x) \ge 1/6$ for all $x \in (0,\pi/2)\cup(3\pi/2, 2\pi)$ and $\lim_{x\to 0} f(x) = 1/4 > 1/6$, hence

$$\int_{\frac{\pi}{m}}^{2\pi-\frac{\pi}{m}} f(x)\,\mathrm{d}x = \int_0^{2\pi} f(x)\,\mathrm{d}x - \int_0^{\frac{\pi}{m}} f(x)\,\mathrm{d}x - \int_{2\pi-\frac{\pi}{m}}^{2\pi} f(x)\,\mathrm{d}x \le \int_0^{2\pi} f(x)\,\mathrm{d}x - \frac{2\pi}{m}\cdot\frac{1}{6}\,.$$

Finally, a numerical calculation gives $\max_{x\in[0,2\pi)}\{|f'(x)|\} < \frac{1}{8}$. To sum up,

$$
\begin{aligned}
&\sum_{j=1}^{m-1} \frac{1-\cos\gamma_j}{(5-4\cos\gamma_j)\ln(5-4\cos\gamma_j)} \\
&\qquad \le \frac{m}{2\pi}\left(\int_0^{2\pi} \frac{1-\cos x}{(5-4\cos\gamma_j)\ln(5-4\cos x)}\,\mathrm{d}x - \frac{2\pi}{m}\cdot\frac{1}{6} + (m-1)\frac{1}{8}\cdot\frac{\pi^2}{m^2}\left(1+\frac{1}{36}\right)\right),
\end{aligned}
$$

whence we obtain the sought formula (35). □

Lemma 5.8. *It holds*

$$\sum_{j=1}^{m-1} \frac{\cos\gamma_j}{5-4\cos\gamma_j} \le \frac{m}{6} + \frac{5}{6}\,. \tag{37}$$

Proof. If we define $\gamma_m := 2\pi$, we can write

$$\sum_{j=1}^{m-1} \frac{\cos\gamma_j}{5-4\cos\gamma_j} = \frac{m}{2\pi}\sum_{j=1}^{m} \frac{2\pi}{m}\frac{\cos\gamma_j}{5-4\cos\gamma_j} - 1\,.$$

The sum $\sum_{j=1}^m \frac{2\pi}{m} f(\gamma_j)$ for $f(\gamma) := \frac{\cos\gamma}{5-4\cos\gamma}$ will be calculated in a similar way as in the proof of Lemma 5.7. Namely, it is a Riemann sum of the function f with respect to the tagged partition

$$\frac{\pi}{m} = x_0 < x_1 < \ldots < x_{m-1} < x_m = 2\pi + \frac{\pi}{m}\,, \qquad \text{where } x_j = \frac{2\pi}{m}\left(j+\frac{1}{2}\right),$$

of interval $\left[\frac{\pi}{m}, 2\pi+\frac{\pi}{m}\right]$. Following the steps of the proof of Lemma 5.7, we obtain

$$
\begin{aligned}
\sum_{j=1}^{m} f(\gamma_j) &\le \frac{m}{2\pi}\left(\int_{\frac{\pi}{m}}^{2\pi+\frac{\pi}{m}} f(x)\,\mathrm{d}x + (m-1)\max_{x\in[0,2\pi)}\{|f'(x)|\}\frac{\pi^2}{m^2}\left(1+\frac{1}{36}\right) + \frac{\pi^2}{m^2}\max_{x\in[2\pi,2\pi+\pi/m)}\{|f'(x)|\}\right) \\
&< \frac{m}{2\pi}\left(\int_{\frac{\pi}{m}}^{2\pi+\frac{\pi}{m}} f(x)\,\mathrm{d}x + m\cdot\max_{x\in[0,2\pi)}\{|f'(x)|\}\frac{\pi^2}{m^2}\left(1+\frac{1}{36}\right)\right).
\end{aligned}
$$

With regard to the properties of cos, we find

$$\int_{\frac{\pi}{m}}^{2\pi+\frac{\pi}{m}} \frac{\cos x}{5-4\cos x}\,\mathrm{d}x = 2\int_0^{\pi} \frac{\cos x}{5-4\cos x}\,\mathrm{d}x = 2\left[-\frac{x}{4} + \frac{5}{6}\arctan\left(3\tan\frac{x}{2}\right)\right]_0^{\pi} = \frac{\pi}{3}\,.$$

Furthermore,

$$\max_{x\in[0,2\pi)}\{|f'(x)|\} = \frac{5}{2}\cdot\frac{\sqrt{10\sqrt{153}-11}}{(15-\sqrt{153})^2} < \frac{9}{8}\,.$$

To sum up,

$$\sum_{j=1}^{m-1} \frac{\cos\gamma_j}{5-4\cos\gamma_j} \le \frac{m}{2\pi}\left(\frac{\pi}{3} + \frac{9}{8}\cdot\frac{\pi^2}{m}\left(1+\frac{1}{36}\right)\right) - 1 = \frac{m}{6} + \frac{\pi}{2}\left(1+\frac{1}{36}\right)\frac{9}{8} - 1 < \frac{m}{6} + \frac{5}{6}\,.$$

□

Proposition 5.9. *For all $m \ge 4$, it holds*

$$\sum_{k=2m}^{+\infty} |g(k)| < \frac{A}{2\pi}m + 1\,, \tag{38}$$

where

$$A := \int_0^{2\pi} \frac{1-\cos x}{(5-4\cos x)\ln(5-4\cos x)}\,\mathrm{d}x \approx 0.909\,. \tag{39}$$

Proof. Recall that

$$\sum_{k=2m}^{+\infty} |g(k)| < \frac{1}{2(m-1)} \cdot \frac{1}{1-\frac{2\ln 3}{3m}} \left(\frac{2m}{1-\frac{\ln 3}{m}} \sum_{j=1}^{m-1} \frac{1-\cos\gamma_j}{(5-4\cos\gamma_j)\ln(5-4\cos\gamma_j)} + \sum_{j=1}^{m-1} \frac{\cos\gamma_j}{5-4\cos\gamma_j} \right) .$$

cf. formula (32). If we estimate the sums using inequalities (35) and (37), we obtain

$$\begin{aligned} &\sum_{k=2m}^{+\infty} |g(k)| - \frac{A}{2\pi} m - 1 \\ &< \frac{1}{2(m-1)} \cdot \frac{1}{1-\frac{2\ln 3}{3m}} \left(\frac{2m}{1-\frac{\ln 3}{m}} \left(\frac{m}{2\pi} A - \frac{1}{6} + \frac{m-1}{m} \cdot \frac{\pi}{16} \left(1 + \frac{1}{36} \right) \right) + \frac{m}{6} + \frac{5}{6} \right) - \frac{A}{2\pi} m - 1 . \end{aligned}$$

A numerical integration gives $A \approx 0.909 \in (0.9, 0.91)$. For such value of A, the expression above is negative for all $m \geq 4$; i.e.,

$$\sum_{k=0}^{+\infty} |g(k)| - \frac{A}{2\pi} m - 1 < 0 \qquad \text{for all } m \geq 4 .$$

□

6 Main result

Theorem 6.1. *For every $m \geq 5$, the m-bonacci word is c-balanced with*

$$c = \lfloor \kappa m \rfloor + 12,$$

where $\kappa = \frac{2}{\pi} \int_0^{2\pi} \frac{1-\cos x}{(5-4\cos x)\ln(5-4\cos x)} \mathrm{d}x \approx 0.58$.

Proof. In Propositions 5.3 and 5.9 we showed

$$\sum_{k=0}^{2m-1} |g(k)| < \frac{5}{4} \qquad \text{and} \qquad \sum_{k=2m}^{+\infty} |g(k)| < \frac{A}{2\pi} m + 1 \quad \text{for all } m \geq 4 ;$$

therefore,

$$\sum_{k=0}^{+\infty} |g(0,k)| < \tfrac{9}{4} + \frac{A}{2\pi} m , \tag{40}$$

where $A = \int_0^{2\pi} \frac{1-\cos x}{(5-4\cos x)\ln(5-4\cos x)} \mathrm{d}x \approx 0.909$.

Having this bound in hand, we can use Corollary 2.4 to estimate the balance constant of letter 0 by

$$c_0 \leq 2 \sum_{k=0}^{+\infty} |g(0,k)| \leq \tfrac{9}{2} + \frac{A}{\pi} m .$$

Since $\frac{9}{2} + \frac{A}{\pi} m \leq 2^{m-1} - 3$ for any $m \geq 5$, the assumption of Proposition 4.1 is fulfilled and thus the m-bonacci word is c-balanced with

$$c = 2c_0 + 3 \leq 3 + 4 \sum_{k=0}^{+\infty} |g(0,k)| \leq 12 + \frac{2A}{\pi} m ,$$

which proves the theorem.

□

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A On eigenvalues of M

In this section we examine the eigenvalues of the matrix of substitution. In particular, we estimate their absolute values and arguments. Such information is essential for estimating the sums $\sum_{k=0}^{2m-1}|g(0,k)|$ and $\sum_{k=2m}^{+\infty}|g(0,k)|$ in Section 5.

Let us recall that the eigenvalues of the matrix of substitution M are zeros of its characteristic polynomial $p(x) = x^m - x^{m-1} - x^{m-2} - \ldots - x - 1$. The following observation will make further calculations substantially simpler.

Observation 4. *Every zero of the polynomial $p(x)$ is a root of the equation*

$$x^m(2-x)=1\,. \tag{41}$$

Proof. For every $x\neq 1$, we can write

$$p(x)=x^m-\frac{x^m-1}{x-1}=\frac{x^{m+1}-2x^m+1}{x-1}\,. \tag{42}$$

In particular, $p(\beta_j)=0$ implies $\beta_j^{m+1}-2\beta_j^m+1=0$, whence β_j is a root of equation (41). $\square$

At first we derive a fine estimate on β, which is needed for calculating the sum $\sum_{k=0}^{2m-1}|g(0,k)|$.

Lemma A.1. *The dominant eigenvalue $\beta>1$ of the matrix of substitution M obeys the inequalities*

$$\frac{1}{2^m-\frac{m}{2}}<2-\beta<\frac{1}{2^m-\frac{m+1}{2}}\,. \tag{43}$$

Proof. Observation 4 implies $\beta^m(2-\beta)=1$, hence $\beta<2$. Let us set $x_0:=2-\beta$. Obviously, x_0 is a root of the polynomial

$$q(x)=(2-x)^m\cdot x-1\,.$$

Since $\beta\in(1,2)$, necessarily $x_0\in(0,1)$. It holds $q'(x)=(2-x)^{m-1}(2-x-mx)$, therefore, q grows in $[0,2/(m+1)]$ and decreases in $[2/(m+1),1]$. Since $q(0)=-1$ and $q(1)=0$, the root x_0 belongs to the interval $(0,2/(m+1))$, in which q grows. Consequently, proving inequalities (43) consists in showing that

$$q\left(\frac{1}{2^m-\frac{m}{2}}\right)<0<q\left(\frac{1}{2^m-\frac{m+1}{2}}\right)\,.$$

Let us start with the estimate of $2-\beta$ from above. We have

$$q\left(\frac{1}{2^m-\frac{m+1}{2}}\right)=\left(2-\frac{1}{2^m-\frac{m+1}{2}}\right)^m\frac{1}{2^m-\frac{m+1}{2}}-1=\left(1-\frac{1}{2^{m+1}-(m+1)}\right)^m\frac{1}{1-\frac{m+1}{2^{m+1}}}-1\,.$$

Since $(1+x)^m>1+mx$ for all $x\in(-1,1)$, it holds

$$q\left(\frac{1}{2^m-\frac{m+1}{2}}\right)>\frac{1-\frac{m}{2^{m+1}-(m+1)}}{1-\frac{m+1}{2^{m+1}}}-1=\frac{-\frac{m}{2^{m+1}-(m+1)}+\frac{m+1}{2^{m+1}}}{1-\frac{m+1}{2^{m+1}}}=\frac{2^{m+1}-(m+1)^2}{\left[2^{m+1}\left(1-\frac{m+1}{2^{m+1}}\right)\right]^2}\geq 0$$

for all $m\geq 3$. Hence, $q\left(1/(2^m-\frac{m+1}{2})\right)>0$ for all $m\geq 3$. If $m=2$, the statement can be proved in the same way, just we use the exact expression $(1+x)^2=1+2x+x^2$ instead of the estimate $(1+x)^m>1+mx$.

Let us proceed to the extimate of $2-\beta$ from below.

$$q\left(\frac{1}{2^m-\frac{m}{2}}\right)=\left(2-\frac{1}{2^m-\frac{m}{2}}\right)^m\frac{1}{2^m-\frac{m}{2}}-1=\frac{1}{1-\frac{m}{2^{m+1}}}\left[\left(1-\frac{1}{2^{m+1}-m}\right)^m-\left(1-\frac{m}{2^{m+1}}\right)\right]\,.$$

For all $x\in(-1,0)$, it holds $(1+x)^m<1+mx+\binom{m}{2}x^2$; therefore,

$$\begin{aligned}\left(1-\frac{1}{2^{m+1}-m}\right)^m-\left(1-\frac{m}{2^{m+1}}\right)&<1-\frac{m}{2^{m+1}-m}+\frac{m(m-1)}{2(2^{m+1}-m)^2}-1+\frac{m}{2^{m+1}}\\&=\frac{m}{2(2^{m+1}-m)^2}\left(-2^{m+2}+2m+m-1+2^{m+2}-4m+\frac{m^2}{2^m}\right)\\&=\frac{m}{2(2^{m+1}-m)^2}\left(-1-m+\frac{m^2}{2^m}\right)<0\end{aligned}$$

for all $m\geq 2$. Hence $q\left(1/(2^m-\frac{m}{2})\right)<0\,.$ $\square$

Now we proceed to the eivenvalues β_j for $j = 1, \dots, m-1$. For the sake of convenience let us set $B_j := |\beta_j|$ and $\gamma_j := \arg(\beta_j)$, i.e.,

$$\beta_j = B_j \mathrm{e}^{\mathrm{i}\gamma_j} \qquad \text{for all } j = 1, \dots, m-1\,.$$

Lemma A.2. *It holds*

$$|\beta_j| < 1 - \frac{\ln(5 - 4\cos\gamma_j)}{2m}\left(1 - \frac{\ln 3}{m}\right) \tag{44}$$

for all $j = 1, \dots, m-1$.

Proof. Since the value $\beta_j = B_j \mathrm{e}^{\mathrm{i}\gamma_j}$ is a solution of equation (41), necessarily

$$\left|B_j^m \mathrm{e}^{\mathrm{i}m\gamma_j}(2 - B_j \mathrm{e}^{\mathrm{i}\gamma_j})\right|^2 = 1\,.$$

Hence

$$B_j^{2m}\left(4 - 4B_j\cos\gamma_j + B_j^2\right) = 1\,. \tag{45}$$

Note that if $m \gg 1$, then obviously $B_j \approx 1$. Therefore, equation (45) can be expressed approximately as

$$B_j^{2m}\left(4 - 4\cos\gamma_j + 1\right) \approx 1 \qquad \text{for } m \gg 1\,.$$

Consequently, for $m \gg 1$ we have

$$\begin{aligned} B_j \approx \frac{1}{\sqrt[2m]{5 - 4\cos\gamma_j}} = \mathrm{e}^{-\frac{\ln(5-4\cos\gamma_j)}{2m}} &\approx \left[\left(1 + \frac{1}{2m}\right)^{\frac{1}{2m}}\right]^{-\frac{\ln(5-4\cos\gamma_j)}{2m}} \\ &= \left(1 + \frac{1}{2m}\right)^{-\ln(5-4\cos\gamma_j)} \approx 1 - \frac{\ln(5 - 4\cos\gamma_j)}{2m}\,. \end{aligned} \tag{46}$$

With regard to this approximation, let us set

$$B_j = 1 - \frac{\ln(5 - 4\cos\gamma_j)}{2m}(1 + \delta_j)\,, \tag{47}$$

for all m, where δ_j compensates the error of the approximation (46). Comparing the statement (44) with the definition of δ_j, we shall prove that

$$\delta_j > -\frac{\ln 3}{m} \qquad \text{for all } j = 1, \dots, m-1\,.$$

We proceed by contradiction. Let there be a $j \in \{1, \dots, m-1\}$ such that $\delta_j \leq -\frac{\ln 3}{m}$. (Note that necessarily $\delta_j > -1$, because β_j's are of moduli less than one.) For all $x > \alpha > 1$, it holds

$$\frac{1}{\left(1 - \frac{\alpha}{x}\right)^x} = \left(1 + \frac{\alpha}{x - \alpha}\right)^x = \left[\left(1 + \frac{1}{\frac{x}{\alpha} - 1}\right)^{\frac{x}{\alpha} - 1}\right]^{\frac{\frac{x}{\alpha}}{\frac{x}{\alpha} - 1}} < \mathrm{e}^{\frac{\frac{x}{\alpha}}{\frac{x}{\alpha} - 1}} = \left(\mathrm{e}^{\alpha}\right)^{1 + \frac{\alpha}{x - \alpha}}\,.$$

Since $B_j = 1 - \frac{\alpha}{x}$ for $x = 2m$ and $\alpha = (1 + \delta_j)\ln(5 - 4\cos\gamma_j)$, we have

$$\frac{1}{B_j^{2m}} < \left(\mathrm{e}^{(1+\delta_j)\ln(5-4\cos\gamma_j)}\right)^{1 + \frac{(1+\delta_j)\ln(5-4\cos\gamma_j)}{2m - (1+\delta_j)\ln(5-4\cos\gamma_j)}} = (5 - 4\cos\gamma_j)^{(1+\delta_j)\left(1 + \frac{(1+\delta_j)\ln(5-4\cos\gamma_j)}{2m - (1+\delta_j)\ln(5-4\cos\gamma_j)}\right)}\,.$$

Our assumption on δ_j implies $\delta_j < 0$, therefore

$$\frac{(1 + \delta_j)\ln(5 - 4\cos\gamma_j)}{2m - (1 + \delta_j)\ln(5 - 4\cos\gamma_j)} \leq \frac{\ln(5 - 4\cos\gamma_j)}{2m - \ln(5 - 4\cos\gamma_j)}\,;$$

hence

$$\frac{1}{B_j^{2m}} < (5-4\cos\gamma_j)^{(1+\delta_j)\left(1+\frac{\ln(5-4\cos\gamma_j)}{2m-\ln(5-4\cos\gamma_j)}\right)}\,. \tag{48}$$

At the same time we have from equation (45)

$$\begin{aligned}\frac{1}{B_j^{2m}} = 4-4B_j\cos\gamma_j+B_j^2 &= 5-4\cos\gamma_j+(1-B_j)(4\cos\gamma_j-2)+(1-B_j)^2\\ &> 5-4\cos\gamma_j+(1-B_j)(4\cos\gamma_j-2)\,. \end{aligned}\tag{49}$$

Putting inequalities (48) and (49) together, we get

$$(5-4\cos\gamma_j)^{(1+\delta_j)\left(1+\frac{\ln(5-4\cos\gamma_j)}{2m-\ln(5-4\cos\gamma_j)}\right)} > 5-4\cos\gamma_j+(1-B_j)(4\cos\gamma_j-2)\,;$$

hence

$$(5-4\cos\gamma_j)^{\delta_j+(1+\delta_j)\frac{(1+\delta_j)\ln(5-4\cos\gamma_j)}{2m-(1+\delta_j)\ln(5-4\cos\gamma_j)}} > 1+(1-B_j)\frac{4\cos\gamma_j-2}{5-4\cos\gamma_j}\,.$$

This gives, with regard to equation (47),

$$\mathrm{e}^{\left(\delta_j+(1+\delta_j)\frac{\ln(5-4\cos\gamma_j)}{2m-\ln(5-4\cos\gamma_j)}\right)\ln(5-4\cos\gamma_j)}-1 > \frac{\ln(5-4\cos\gamma_j)}{2m}(1+\delta_j)\frac{4\cos\gamma_j-2}{5-4\cos\gamma_j}\,. \tag{50}$$

Since $\delta_j \le -\frac{\ln 9}{2m} \le -\frac{\ln(5-4\cos\gamma_j)}{2m}$ by assumption, it holds

$$\delta_j+(1+\delta_j)\frac{\ln(5-4\cos\gamma_j)}{2m-\ln(5-4\cos\gamma_j)} \le 0\,,$$

therefore, the exponent in (50) is negative (or zero). Moreover, a simple analysis of the exponent, using the fact $\delta_j > -1$, leads to the inequality

$$\left(\delta_j+(1+\delta_j)\frac{\ln(5-4\cos\gamma_j)}{2m-\ln(5-4\cos\gamma_j)}\right)\ln(5-4\cos\gamma_j) \ge -\ln 9 \qquad \text{for all } \gamma_j\in\mathbb{R}\,.$$

The convexity of the exponential function implies

$$\mathrm{e}^x-1 < \frac{\mathrm{e}^b-1}{b}x$$

for all $b < x \le 0$. Therefore, the left hand side of (50) obeys

$$\begin{aligned}&\mathrm{e}^{\left(\delta_j+(1+\delta_j)\frac{\ln(5-4\cos\gamma_j)}{2m-\ln(5-4\cos\gamma_j)}\right)\ln(5-4\cos\gamma_j)}-1\\ &\qquad< \frac{1-\mathrm{e}^{-\ln 9}}{\ln 9}\left(\delta_j+(1+\delta_j)\frac{\ln(5-4\cos\gamma_j)}{2m-\ln(5-4\cos\gamma_j)}\right)\ln(5-4\cos\gamma_j)\\ &\qquad= \frac{8}{9\ln 9}\left(\delta_j+(1+\delta_j)\frac{\ln(5-4\cos\gamma_j)}{2m-\ln(5-4\cos\gamma_j)}\right)\ln(5-4\cos\gamma_j)\,.\end{aligned}$$

Inequality (50) together with this estimate imply

$$\frac{8}{9\ln 9}\left(\delta_j+(1+\delta_j)\frac{\ln(5-4\cos\gamma_j)}{2m-\ln(5-4\cos\gamma_j)}\right)\ln(5-4\cos\gamma_j) > \frac{\ln(5-4\cos\gamma_j)}{2m}(1+\delta_j)\frac{4\cos\gamma_j-2}{5-4\cos\gamma_j}\,.$$

We divide both sides by $\ln(5-4\cos\gamma_j)$, which is allowed due to $\gamma_j \neq 0$ (recall that $\beta_j \notin (0,+\infty)$ for all $j=1,\ldots,m-1$); hence

$$\delta_j+(1+\delta_j)\frac{\ln(5-4\cos\gamma_j)}{2m-\ln(5-4\cos\gamma_j)} > \frac{9\ln 9}{8}\cdot\frac{1+\delta_j}{2m}\cdot\frac{4\cos\gamma_j-2}{5-4\cos\gamma_j}\,. \tag{51}$$

For all $\gamma_j \in \mathbb{R}$, $\ln(5-4\cos\gamma_j) \le \ln 9$ and

$$\frac{4\cos\gamma_j - 2}{5-4\cos\gamma_j} = -1 + \frac{3}{5-4\cos\gamma_j} \ge -1 + \frac{3}{9} = -\frac{2}{3}\,;$$

therefore, with regard to inequality (51),

$$\delta_j + (1+\delta_j)\frac{\ln 9}{2m-\ln 9} > \frac{9\ln 9}{8}\cdot\frac{1+\delta_j}{2m}\cdot\frac{-2}{3} = -\frac{3\ln 9}{8m}(1+\delta_j)\,.$$

Consequently,

$$\left(1+\frac{1}{2m-\ln 9}+\frac{3}{8m}\right)\delta_j > -\frac{1}{2m-\ln 9}-\frac{3}{8m}\,;$$

hence

$$\delta_j \ge -\frac{\frac{1}{2m-\ln 9}+\frac{3}{8m}}{1+\frac{1}{2m-\ln 9}+\frac{3}{8m}}\,.$$

This is a contradiction with the assumption $\delta_j \le -\frac{\ln 3}{m}$, because

$$-\frac{\ln 3}{m} < -\frac{\frac{1}{2m-\ln 9}+\frac{3}{8m}}{1+\frac{1}{2m-\ln 9}+\frac{3}{8m}} \qquad \text{for all } m \ge 2\,.$$

□

Lemma A.3. *The arguments of β_j satisfy*

$$\gamma_j \in \left(j\frac{2\pi}{m}-\frac{\pi}{6m}, j\frac{2\pi}{m}+\frac{\pi}{6m}\right) \tag{52}$$

for all $j = 1,\dots,m-1$.

Proof. Equation (41) has $m+1$ solutions, namely 1, β and $\beta_1,\dots,\beta_{m-1}$. Therefore, it suffices to show that every sector

$$\mathcal{S}_j := \left\{Be^{i\gamma}\,\middle|\, B>0\,,\, \gamma \in \left(j\frac{2\pi}{m}-\frac{\pi}{6m}, j\frac{2\pi}{m}+\frac{\pi}{6m}\right)\right\} \qquad \text{for } j=1,\dots,m-1$$

contains exactly one solution of equation (41).

Let

$$\beta = Be^{i\gamma}$$

be a solution of (41), i.e.,

$$B^m e^{im\gamma}\left(2-Be^{i\gamma}\right) = 1\,.$$

Hence

$$m\gamma = -\arg\left(2-Be^{i\gamma}\right)+2j\pi \qquad \text{for a certain } j\in\mathbb{Z}\,. \tag{53}$$

We can obviously assume $j \in \{0,1,\dots,m-1\}$ without loss of generality. Since the solutions 1 and β of equation (53) are obtained for $\gamma=0$, and, therefore, for $j=0$, we prove the statement in two steps: 1. We demonstrate that equation (55) has exactly one solution for every $j=1,\dots,m-1$. 2. We show that the solution corresponding to j belongs to the sector S_j for every $j=1,\dots,m-1$.

It holds

$$2-Be^{i\gamma} = 2-B\cos\gamma - iB\sin\gamma\,,$$

hence

$$\tan(\arg(2-Be^{i\gamma})) = \frac{-B\sin\gamma}{2-B\cos\gamma} = \frac{-\sin\gamma}{\frac{2}{B}-\cos\gamma}\,.$$

Furthermore, $B < 1$ implies $2 - B\cos\gamma > 0$, hence

$$\arg(2 - B\mathrm{e}^{\mathrm{i}\gamma}) \in (-\pi/2, \pi/2)\,, \tag{54}$$

i.e., we can write

$$\arg\left(2 - B\mathrm{e}^{\mathrm{i}\gamma}\right) = \arctan\frac{-\sin\gamma}{\frac{2}{B} - \cos\gamma}\,.$$

To sum up, equation (53) is equivalent to

$$m\gamma - \arctan\frac{\sin\gamma}{\frac{2}{B} - \cos\gamma} = 2j\pi\,. \tag{55}$$

For every $j = 1, \ldots, m-1$, the left hand side $L(\gamma) = m\gamma - \arctan\frac{\sin\gamma}{\frac{2}{B}-\cos\gamma}$ of equation (55), regarded as a function of γ with a fixed $B < 1$, is continuous and satisfies

$$0 = L(0) < 2j\pi < 2m\pi = L(2\pi)\,.$$

Also, a simple calculation gives

$$L'(\gamma) = m - \frac{\frac{2}{B}\cos\gamma - 1}{\left(\frac{2}{B}\right)^2 - 2\cdot\frac{2}{B}\cos\gamma + 1} > m - \frac{1}{\frac{2}{B} - 1} > m - 1 > 0\,.$$

Consequently, equation (55) has indeed exactly one solution for every $j = 1, \ldots, m-1$. The solution satisfies $m\gamma - 2j\pi \in (-\pi/2, \pi/2)$. With regard to the numbering (3), we conclude that

$$\gamma_j \in \left(\frac{2j\pi}{m} - \frac{\pi}{2m}, \frac{2j\pi}{m} + \frac{\pi}{2m}\right)\,.$$

Now we improve this estimate in order to prove $\gamma_j \in \mathcal{S}_j$. Since $2/B_j > 2$ for all $j = 1, \ldots, m-1$, we have

$$\left|\frac{-\sin\gamma_j}{\frac{2}{B_j} - \cos\gamma_j}\right| \le \left|\frac{\sin\gamma_j}{2 - \cos\gamma_j}\right|\,.$$

It is easy to show that

$$\left|\frac{\sin\gamma}{2 - \cos\gamma}\right| \le \frac{1}{\sqrt{3}} \qquad \text{for all } \gamma \in \mathbb{R}\,,$$

hence

$$\left|\arctan\frac{\sin\gamma_j}{\frac{2}{B_j} - \cos\gamma_j}\right| \le \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}\,. \tag{56}$$

By substituting estimate (56) into equation (55), we obtain statement (52). $\square$